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Approximation numbers = singular values

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Dedicated to Des Evans on his 65th birthday

Abstract

This paper introduces a generalisation of the notion of singular value for Hilbert space operators to more general Banach spaces. It is shown that for a simple integral operator of Hardy type the singular values are the eigenvalues of a non-linear Sturm-Liouville equation and coincide with the approximation numbers of the operator. Finally, asymptotic formulas for the singular numbers are deduced.

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0. Introduction

A natural way of measuring the 'degree of compactness' for an operator in Hilbert space is to study its singular values, in particular their asymptotic behaviour. For operators in more general Banach spaces many replacements for the singular values have been suggested, but among these the *approximation numbers* seem to have the widest acceptance. It seems, however, that the idea of singular values is too attractive to be dismissed out of hand. While singular values are the (square roots of the) eigenvalues of a linear operator, the directly corresponding concept for operators in more general Banach spaces are 'eigenvalues' of a non-linear equation. Even the existence of these more general singular values may therefore be in question.

In this note we shall consider the Hardy operator defined by

$$Tf(x) = v(x)\int_a^x uf, \tag{0.1}$$

considered as an operator on $L^p(a,b)$, where $1<p<\infty$, (a,b) is a bounded interval and u, v are given functions. This operator is related to certain embedding operators of weighted Sobolev spaces, but we will not discuss such matters here. We shall see that in this case the generalised singular values do exist, and coincide with the approximation numbers. The singular values are simply related to the eigenvalues of a certain non-linear differential equation, which for $p=2$ reduces to a Sturm–Liouville equation. This will, with little effort, enable us to find asymptotic formulas for the approximation numbers.

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Various aspects of our subject matter have been dealt with repeatedly in the literature. Some recent books and papers connected with the subject matter of these notes are [3,4,9,7,10,8,11], where further references to the literature may also be found.

Section 1 introduces our generalised singular values and deduces the non-linear Sturm–Liouville eigenvalue problem which defines the singular values of our operator T. Section 2 is devoted to proving the existence of eigenvalues and oscillation properties of the eigenfunctions for this equation, and the main result of Section 3 is the equality of the approximation numbers and the eigenvalues. In Section 4 one-term asymptotics for the eigenvalues are given, and in the last section a better estimate of the error is deduced, assuming some smoothness of a particular combination of the functions u and v.

Remark 0.1. This paper is based on the report [2]. After completion of the paper it was pointed out to the author that results on the asymptotic behaviour of the eigenvalues of the non-linear Sturm–Liouville equation may also be found in the paper [5].

1. Singular values

Proposition 1.1. *Let B_1, B_2 be Banach spaces, with B_1 reflexive, and suppose $T : B_1 \to B_2$ is a compact operator. Then there exists a non-zero element $f \in B_1$ such that $\|T\| = \|Tf\|/\|f\|$.*

This is well known. It means in particular that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\|T(f+\varepsilon\varphi)\|}{\|f+\varepsilon\varphi\|}|_{\varepsilon=0}=0$, if the derivative (considering ε as a real variable) exists at $\varepsilon = 0$. It is well known, see e.g. [13, Section 5.4], that the Gateaux derivative of the norm of B_1 exists at $f \neq 0$ if the dual of B_1 is *strictly convex*. Strictly convex spaces include all L^p-spaces with $1 < p < \infty$. The derivative $(\mathrm{d}/\mathrm{d}\varepsilon)\|f+\varepsilon\varphi\||_{\varepsilon=0}$ then equals $Re\langle L_1(f), \varphi\rangle$, where $L_1(f)$ is the unique *dual vector* to f, i.e., $L_1(f)$ is a unit vector in the dual B_1' of B_1 with the property $\langle L_1(f), f\rangle = \|f\|$.

We now assume that B_1' and the dual B_2' of B_2 are strictly convex, and denote by L_2 the map taking vectors in $B_2\setminus\{0\}$ to their dual vectors in B_2'.

Proposition 1.2. *If $T \neq 0$ and f is an extremal according to Proposition* 1.1, *and B_1', B_2' are strictly convex, then f satisfies the 'Euler equation'*

$$T^*L_2(Tf) = \mu L_1(f), \tag{1.1}$$

where $\mu = \|Tf\|/\|f\|$.

Conversely, if $f \neq 0$ satisfies (1.1) *for some μ, then $\mu > 0$ and $\|Tf\| = \mu\|f\|$.*

Proof. We have $\mathrm{d}/\mathrm{d}\varepsilon\|T(f+\varepsilon\varphi)\| = \mathrm{Re}\langle L_2(T(f+\varepsilon\varphi)), T\varphi\rangle$ which equals $\mathrm{Re}\langle T^*L_2(T(f+\varepsilon\varphi)), \varphi\rangle$, and $\mathrm{d}/\mathrm{d}\varepsilon\|f+\varepsilon\varphi\| = \langle L_1(f+\varepsilon\varphi), \varphi\rangle$ so that (1.1) follows from $\mathrm{d}/\mathrm{d}\varepsilon\frac{\|T(f+\varepsilon\varphi)\|}{\|f+\varepsilon\varphi\|}|_{\varepsilon=0} = 0$ for all $\varphi \in L^p(a, b)$.

Conversely, $\|Tf\| = \langle L_2(Tf), Tf\rangle = \langle T^*L_2(Tf), f\rangle$, and using (1.1) this equals $\mu\langle L_1(f), f\rangle = \mu\|f\|$, so that $\mu = \|Tf\|/\|f\| \geqslant 0$. Note that if $\mu = 0$, then $Tf = 0$, but if we define $L_2(0) = 0$, then the calculation is correct also in this case. □

Thus the norm of T is the largest 'eigenvalue' μ_1 of (1.1), which may be viewed as an Euler equation for maximising $\|Tf\|$ under the side condition $\|f\| = 1$. By Proposition 1.1 this largest eigenvalue exists, and this and any additional eigenvalues > 0 we call the *singular values* for the operator T. Note that this agrees with standard terminology for the case when B_1, B_2 are Hilbert spaces. Here is another fact reminiscent of the Hilbert space case.

Proposition 1.3. *If B_1, B_2 and their duals are all strictly convex, then the singular values of T coincide with those of T^*.*

Proof. First note that if $f \in B_j$ is a unit vector and $g \in B_j'$ its dual vector, then f is the dual vector of g. Thus, if L_j' maps a vector $\neq 0$ in B_j' to its dual vector, then $L_j'(L_j(f)) = f/\|f\|$.

If now $f \neq 0$ satisfies (1.1) for some $\mu > 0$, then $g = L_2(Tf) \neq 0$ and $\mu L_1(f) = T^*g$ so that $L_1'(T^*g) = f/\|f\|$. Furthermore, $L_2'(g) = Tf/\|Tf\|$ so that $TL_1'(T^*g) = \|Tf\|/\|f\|L_2'(g) = \mu L_2'(g)$. Thus any singular value for T is a singular value for T^*. The converse also holds since $T^{**} = T$. □

To obtain a quantitative measure of the compactness of the operator T it is customary to introduce the approximation numbers

$$a_n = \inf_{\text{rank } P<n} \|T - P\|, \quad n = 1, 2, \ldots .$$

Thus $a_1 = \|T\| = \mu_1$ and the sequence $a_1, a_2, \ldots$ is decreasing, with limit 0 if T is a compact operator which can be approximated by operators of finite rank.

Another measure of the compactness of T is given by

$$b_n = \sup_{\dim B \geqslant n} \inf_{0 \neq f \in B} \frac{\|Tf\|}{\|f\|}, \quad n = 1, 2, \ldots,$$

where B is a subspace of B_1. These are essentially the so called 'Bernstein widths' of T. Again, it is clear that $b_1 = \|T\| = \mu_1 = a_1$. The following lemma is basic and well known.

Lemma 1.4. $b_n \leqslant a_n$, $n = 1, 2, \ldots$.

Proof. Suppose $P : B_1 \to B_2$ is an operator with rank $P < n$ and B a subspace of B_1 with $\dim B \geqslant n$. Then $\ker P|_B$ is non-trivial, and if g is a unit vector in the kernel we have $Pg = 0$ so that

$$\inf_{f \in B} \frac{\|Tf\|}{\|f\|} \leqslant \|Tg\| = \|Tg - Pg\| \leqslant \|T - P\|.$$

Since P and B are arbitrary with rank $< n$ respectively dimension $\geqslant n$ the lemma follows. □

We now specialise to $B_1 = B_2 = L^p(\Omega)$ where Ω is a measure space and $1 < p < \infty$. Then, if $f \neq 0$ is a vector in $L^p(\Omega)$ the dual vector of f is easily seen to be $\|f\|^{1-p}|f|^{p-2}\overline{f} \in L^{p'}(\Omega)$, where $p' = p/(p-1)$ is the dual exponent of p. It follows that the 'Euler equation' (1.1) in this case becomes

$$T^*(|Tf|^{p-2}\overline{Tf}) = \mu^p|f|^{p-2}\overline{f}, \tag{1.2}$$

where $\mu = \|Tf\|/\|f\|$.

Finally, we specialise to the simple operator T given by (0.1) where $u \in L^{p'}(a, b)$, $v \in L^p(a, b)$ are given functions. By dominated convergence this is clearly a compact operator. An easy calculation shows that $T^*f(x) = u(x)\int_x^b vf$ so that setting $g(x) = \int_a^x uf$ in the Euler equation we have $g(a) = 0$ and $g' = uf$. Multiplying by $\mu^{-p}|u|^{p-2}\overline{u}$ the Euler equation now reads

$$\lambda|u|^p \int_x^b |v|^p|g|^{p-2}\overline{g} = |g'|^{p-2}\overline{g'}, \tag{1.3}$$

where $\lambda = \mu^{-p}$. If $u \neq 0$ *a.e.* we may divide by $|u|^p$ and obtain

$$\lambda \int_x^b |v|^p|g|^{p-2}g = ||u|^{-p'}g'|^{p-2}|u|^{-p'}g'.$$

From this follows that $(|u|^{-p'}g')(b) = 0$, in the sense that $|u|^{-p'}g'$ is continuous and is zero at b. Differentiating we finally obtain

$$\begin{cases} -(||u|^{-p'}g'|^{p-2}|u|^{-p'}g')' = \lambda|v|^q|g|^{p-2}g, \\ g(a) = (|u|^{-p'}g')(b) = 0 \end{cases} \tag{1.4}$$

as our 'non-linear Sturm–Liouville' eigenvalue problem. If u may vanish on sets of positive measure it is more convenient to instead write this as a first-order system. Defining $g(x) = \int_a^x uf$ and $h(x) = \lambda\int_x^b \overline{v}|Tf|^{p-2}Tf = \lambda\int_x^b |v|^p|g|^{p-2}g$

we obtain $g(a)=h(b)=0$, $h'=-\lambda|v|^p|g|^{p-2}g$ and (1.3) now reads $|u|^p h=|g'|^{p-2}g'$ which is easily seen to be equivalent to $g'=|u|^{p'}|h|^{p'-2}h$. We thus obtain an eigenvalue problem of the form

$$\begin{cases} g'=|u|^{p'}|h|^{p'-2}h, \\ h'=-\lambda|v|^p|g|^{p-2}g, \\ g(a)=h(b)=0, \end{cases} \tag{1.5}$$

where λ is real. We may always restrict ourselves to consider only real-valued solutions of (1.5). This is seen as follows. Multiplying the first equation by $\overline{h}$, integrating and then integrating by parts and using the second equation we obtain

$$\int_a^x |u|^{p'}|h|^{p'} = g(x)\overline{h(x)} + \lambda \int_a^x |v|^p|g|^p,$$

so that $g\overline{h}$ is real-valued, thus also g'/g between zeros of g, as is h'/h between zeros of h. It follows that g and h have constant arguments, which are equal modulo π, between each pair of their respective zeros. So, unless g and h have common zeros, the solution (g,h) is a fixed multiple of a real solution. However, if $g(c)=h(c)=0$ we have

$$g(x)=-\lambda^{p'-1}\int_c^x |u(y)|^{p'}\left|\int_c^y |v|^p|g|^{p-2}g\right|^{p'-2}\int_c^y |v|^p|g|^{p-2}g\,\mathrm{d}y$$

so that

$$\|g\|_\infty \leqslant \lambda^{p'-1}\int_I |u|^{p'}\left(\int_I |v|^p\right)^{p-1}\|g\|_\infty,$$

where I is any interval containing c and $\|\cdot\|_\infty$ the corresponding maximum norm. This shows that g, and thus h, is identically zero in a neighbourhood of c. Thus the set of common zeros of g and h is open and closed and thus all of $[a,b]$.

Proposition 1.5. *For real λ any solution of* (1.5) *is a constant multiple of a real-valued solution.*

For $p\neq 2$ the differential equations (1.4), (1.5) are strongly non-linear, and will not satisfy standard conditions for existence and uniqueness of the initial value problem. Nevertheless, the initial value problems do have unique solutions, specifying the values of g and $|u|^{-p'}g'$, respectively, g and h in the initial point. We shall prove this in the next section.

2. Eigenvalues

The problem (1.5) may be viewed as a kind of generalised Sturm–Liouville eigenvalue problem, and our first task will be to prove that there exists an infinite sequence of eigenvalues. In the next section, we will express the approximation numbers for the operator T in terms of the corresponding eigenvalues. To do this the oscillation properties of the eigenfunctions will be crucial, and for the standard Sturm–Liouville case such oscillation properties are often deduced using a Prüfer transform. We will here use a generalised Prüfer type transform, similar to what was done in [14,6].

We need to use a kind of generalised trigonometric functions for this. This could be done using the functions discussed in, e.g., [12,4]. However, we shall find it convenient to use a related set of functions, which have the advantage of a slightly more symmetric definition. Thus we define the pair of functions $\mathrm{s}_p, \mathrm{c}_p$ as the solution of the initial value problem

$$\begin{cases} \mathrm{s}_p'=|\mathrm{c}_p|^{p'-2}\mathrm{c}_p, \\ \mathrm{c}_p'=-|\mathrm{s}_p|^{p-2}\mathrm{s}_p, \\ \mathrm{s}_p(0)=0, \quad \mathrm{c}_p(0)=1. \end{cases} \tag{2.1}$$

The existence of a solution follows from Peano's theorem, and uniqueness follows from the fact that differentiation immediately shows that a solution satisfies the identity

$$(p'-1)|\mathrm{s}_p|^p+|\mathrm{c}_p|^{p'}=1. \tag{2.2}$$

Thus solutions are confined to the curve $f(x, y) = 1$ where $f(x, y) = (p' - 1)|x|^p + |y|^{p'}$. Here grad $f = p'(|x|^{p-2}x, |y|^{p'-2}y)$ does not vanish on the curve, which therefore is of class C^1, and obviously simple and closed. The system (2.1) determines a unique parametrisation of the curve, and therefore the solution $\mathrm{s}_p, \mathrm{c}_p$ is also unique. It is also defined on the whole real axis, and thus periodic, since the only equilibrium point of 2.1 is the origin, which is not on the curve.

In fact, in terms of the functions $\sin_p$ and $\cos_p$ as defined in [4] one easily obtains $\mathrm{s}_p(x) = (p-1)^{1/p} \sin_p((p-1)^{-1/p}x)$ and also $\mathrm{c}_p(x) = |\cos_p((p-1)^{-1/p}x)|^{p-2} \cos_p((p-1)^{-1/p}x)$. The functions $\mathrm{s}_p, \mathrm{c}_p$ are therefore odd and even, respectively, they are periodic with period $2\tilde{\pi}_p := 2\pi_p(p-1)^{1/p}$ where $\pi_p = 2\int_0^1 (1 - t^p)^{-1/p}\, dt$ and the zeros of s_p and c_p are the even and odd integer multiples of $\tilde{\pi}_p/2$, respectively.

We now define a generalised Prüfer transform by defining functions r and φ so that

$$\begin{cases} g = r\,\mathrm{s}_p(\varphi), \\ h = \lambda^{1/p'} r^{p-1} \mathrm{c}_p(\varphi). \end{cases}$$

It is clear that if $\lambda > 0$ this determines $r \geqslant 0$ uniquely, and φ uniquely modulo $2\tilde{\pi}$ except when $g = h = 0$. The map $(r, \varphi) \mapsto (g, h)$ is clearly C^1 with a C^1 local inverse for $r > 0$, so this is quite similar to introducing polar coordinates in the phase plane. Substituting in (1.5) we obtain

$$\begin{cases} r'/r = \lambda^{1/p}(p' - 1)(|v|^p - |u|^{p'}) s_p'(\varphi) c_p'(\varphi), \\ \varphi' = \lambda^{1/p}(|u|^{p'} |\mathrm{c}_p(\varphi)|^{p'} + |v|^p (p' - 1)|\mathrm{s}_p(\varphi)|^p). \end{cases}$$

The second equation is of the form $\varphi'(x) = G(x, \varphi(x), \lambda)$, where G increases pointwise with λ and satisfies a Lipschitz condition

$$|G(x, \varphi, \lambda) - G(x, \psi, \lambda)| \leqslant \lambda^{1/p} U(x) |\varphi - \psi|,$$

with U integrable. Thus the initial value problem for φ has a unique solution continuous in λ and point-wise increasing with λ. Since the first equation is linear in r with an integrable coefficient it follows that the initial value problem for r, φ is uniquely solvable. The following proposition follows.

Proposition 2.1. *The initial value problem for* (1.5) *is uniquely solvable.*

We are primarily interested in the second equation. The initial condition for g allows us to specify $\varphi(a) = 0$, and the final condition for h shows that we must have $\varphi(b)$ equal to an odd multiple of $\tilde{\pi}_p/2$. Note that we have $\varphi' \geqslant \lambda^{1/p} \min(|u|^{p'}, |v|^p) \geqslant 0$. It follows from this, unless $\min(|u|^{p'}, |v|^p) = 0$ *a.e.*, that $\varphi(b) \geqslant \lambda^{1/p} \int_a^b \min(|u|^{p'}, |v|^p) \to \infty$ as $\lambda \to \infty$. We obtain the following theorem.

Theorem 2.2. *Suppose* $\int_a^b \min(|u|^{p'}, |v|^p) > 0$. *Then there is a sequence* $\lambda_1, \lambda_2, \lambda_3, \ldots$ *of eigenvalues tending to infinity for the problem* (1.5), *corresponding to numbers* $\mu_n = \lambda_n^{-1/p}$ *in* (1.1) *tending to zero. The eigenfunctions* g_n *and* h_n *corresponding to* λ_n *are determined up to constant multiples and have precisely* $n - 1$ *interior zeros in* (a, b). *Moreover, the zeros of* g_n *and* h_n *interlace.*

3. Approximation numbers

In this section we shall prove the following theorem.

Theorem 3.1. *The approximation numbers and Bernstein widths for* T *are given by* $a_n = b_n = \mu_n$, $n = 1, 2, 3, \ldots$.

Since we already know that $b_n \leqslant a_n$, this follows from the following two lemmas.

Lemma 3.2. $a_n \leqslant \mu_n$.

Proof. First note that the norm of T is $\mu_1 = \|T\varphi\|/\|\varphi\|$, where φ is the extremal of Proposition 1.1. We denote the eigenfunction for $\mu_n = \lambda_n^{-1/p}$ by (g_n, h_n), the zeros of g_n by $a = s_0, s_1, \ldots, s_{n-1}$ and the zeros of h_n by $d_1, \ldots, d_n = b$.

By the interlacing property we then have $s_{j-1} < d_j < s_j < d_{j+1}$, $j = 1, \dots, n-1$. We now define an operator P on $L^p(a,b)$ of rank $n-1$ by setting

$$Pf(x) = v(x) \sum_{j=1}^{n-1} g(s_j)\chi_j(x),$$

where as before $g(x) = \int_a^x uf$ and χ_j is the characteristic function of the interval $I_j = (d_j, d_{j+1})$. Thus $a_n \leqslant \|T - P\|$, and we must prove that $\|T - P\| \leqslant \mu_n$.

Now, the operator $T - P$ is clearly the direct sum of operators just like T but operating on $L^p(s_{j-1}, d_j)$ or $L^p(d_j, s_j)$, in the latter case with the boundary conditions interchanged. Thus, the eigenfunction for the highest eigenvalue for all these operators is the restriction of (g_n, h_n) to the appropriate interval, since these functions satisfy the Euler equation and the boundary conditions, and have no interior zeros. Thus, if we denote the operator for the interval I_j (we let $I_0 = (a, d_1)$) by T_j, then $\|T_j\| = \mu_n$. Now, for any $\varphi \in L^p(a,b)$ put $\varphi_j = \varphi\chi_j$. Then

$$\begin{aligned}\|T\varphi - P\varphi\|^p &= \sum \|T_j\varphi_j\|^p \\ &\leqslant \sum \|T_j\|^p\|\varphi_j\|^p = \mu_n^p \sum \|\varphi_j\|^p = \mu_n^p\|\varphi\|^p.\end{aligned}$$

We get equality if $u\varphi = g_n'$, so that in fact $\|T - P\| = \mu_n$. $\square$

Lemma 3.3. $\mu_n \leqslant b_n$.

Proof. The lemma follows if we can construct a subspace $B \subset L^p(a,b)$ of dimension n such that $\mu_n\|f\| \leqslant \|Tf\|$ for any $f \in B$.

To this end, put $f_j = f\chi_{(s_{j-1},s_j)}$, $j = 1, \dots, n-1$, with $\chi_{(s_{j-1},s_j)}$ the characteristic function of (s_{j-1}, s_j) and $f = |u|^{p'-2}\overline{u}|h_n|^{p'-2}h_n$ so that $g_n' = uf$, and $f_n = f\chi_{(s_{n-1},d_n)}$. The linear hull B of $f_1, \dots, f_n$ is thus n-dimensional. Note that by construction neither the supports of f_j, $j = 1, 2, \dots$, nor the supports of Tf_j, $j = 1, 2, \dots$, overlap. Furthermore, by the Euler equation f_j satisfies we have $\|Tf_j\| = \mu_n\|f_j\|$. Thus, if $\varphi = \sum x_j f_j$ we obtain

$$\begin{aligned}\mu_n^p\|\varphi\|_p^p &= \mu_n^p \sum |x_j|^p\|f_j\|^p = \sum |x_j|^p\|Tf_j\|^p \\ &= \sum \|T(x_j f_j)\| \\ &= \|T\varphi\|^p.\end{aligned}$$

The lemma is proved. $\square$

4. Asymptotics

To obtain an asymptotic formula for the singular values, thus also for the approximation numbers and Bernstein widths, we modify the Prüfer transform introduced in Section 2. This is done similarly to what was done in [1] for a linear Sturm–Liouville equation, using standard trigonometric functions. Let ω be a function for which $\log\omega$ is absolutely continuous and put

$$\begin{cases} g = r\,\mathrm{s}_p(\varphi), \\ h = \lambda^{1/p'}(\omega r)^{p-1}\mathrm{c}_p(\varphi). \end{cases}$$

The equation for φ is then modified to

$$\varphi' = \lambda^{1/p}(\omega|u|^{p'}|\mathrm{c}_p(\varphi)|^{p'} + \omega^{1-p}|v|^p(p'-1)|\mathrm{s}_p(\varphi)|^p) + \frac{\omega'}{\omega}\,\mathrm{s}_p(\varphi)\mathrm{c}_p(\varphi). \tag{4.1}$$

To simplify (4.1) we would like to choose ω so that $\omega|u|^{p'} = \omega^{1-p}|v|^p$. The latter function would then be $|uv|$, which is obtained for $\omega = |v||u|^{1-p'}$. Note that $uv \in L^1(a,b)$, but that this choice of ω may not be absolutely continuous or non-vanishing. We therefore first pick a number $\delta \in (0,1)$ and define

$$\tilde{\omega} = \max(\delta, \min(1/\delta, |v||u|^{1-p'})).$$

This is to be interpreted as $1/\delta$ wherever u vanishes. Then $\delta \leqslant \tilde{\omega} \leqslant 1/\delta$, and $\tilde{\omega} \to |v||u|^{1-p'}$ *a.e.* where u does not vanish as $\delta \to 0$. Furthermore $\tilde{\omega}|u|^{p'}$ is bounded by $\max(|u|^{p'}, |uv|)$ and $\tilde{\omega}^{1-p}|v|^p$ by $\max(|v|^p, |uv|)$, so these functions converge in $L^1(a, b)$ to $|uv|$ as $\delta \to 0$. Given $\varepsilon > 0$ we may therefore choose $\delta > 0$ so that we have $\|\tilde{\omega}|u|^{p'} - |uv|\|_1 < \varepsilon$ and $\|\tilde{\omega}^{1-p}|v|^p - |uv|\|_1 < \varepsilon$. Here and later $\|\cdot\|_1$ denotes the norm of $L^1(a, b)$.

Next, let $\chi_\sigma = (1/\sigma)\chi_{[0,\sigma]}$ and put $\omega = \tilde{\omega} * \chi_\sigma$; outside $[a, b]$ we set $\tilde{\omega} = 1$. Then $\delta \leqslant \omega \leqslant 1/\delta$ and ω is absolutely continuous. Furthermore, $\omega \to \tilde{\omega}$ *a.e.* as $\sigma \to 0$ by Lebesgue's differentiation theorem. Thus, by dominated convergence $\omega|u|^{p'} \to \tilde{\omega}|u|^{p'}$ and $\omega^{1-p}|v|^p \to \tilde{\omega}^{1-p}|v|^p$ in $L^1(a, b)$ as $\sigma \to 0$. Setting $A = \omega|u|^{p'} - |uv|$ and $B = \omega^{1-p}|v|^p - |uv|$ we may therefore choose σ so that $\|A\|_1 < \varepsilon$ and $\|B\|_1 < \varepsilon$.

For $\lambda = \lambda_n$ we have as before that $\varphi(a) = 0$, $\varphi(b) = (2n-1)/2\,\tilde{\pi}_p$, so integrating (4.1) we obtain

$$\frac{2n-1}{2}\,\tilde{\pi}_p = \lambda_n^{1/p} \int_a^b (|uv| + A|\mathrm{c}_p(\varphi)|^{p'} + B(p'-1)|\mathrm{s}_p(\varphi)|^p) + \int_a^b \frac{\omega'}{\omega}\mathrm{s}_p(\varphi)\mathrm{c}_p(\varphi).$$

Thus, since $\mu_n = \lambda_n^{-1/p}$, we have $n\mu_n = \mathcal{O}(1)$ as $n \to \infty$, so that

$$|n\mu_n\tilde{\pi}_p - \|uv\|_1| \leqslant \mathcal{O}(1/n) + \|A\|_1 + \|B\|_1.$$

Since $\|A\|_1 + \|B\|_1 < 2\varepsilon$ and $\varepsilon > 0$ is arbitrary we finally obtain the following theorem.

Theorem 4.1. *For the sequence of singular values $\mu_1, \mu_2, \ldots$ of T one has*

$$\mu_n = \frac{\|uv\|_1}{n\tilde{\pi}_p} + \mathrm{o}(n^{-1}) \quad \textit{as } n \to \infty.$$

5. Improved asymptotics

If we assume some smoothness of the coefficients we can improve Theorem 4.1 and estimate the error in the asymptotic formula. The smoothness assumption is simply that choosing the function ω of the previous section to be $\omega = |v||u|^{1-p'}$ we have ω'/ω integrable or, equivalently, $\log(|v|^p/|u|^{p'})$ is absolutely continuous. With this choice of ω Eq. (4.1) becomes

$$\varphi' = \lambda^{1/p}|uv| + \frac{\omega'}{\omega}\,\mathrm{s}_p(\varphi)\mathrm{c}_p(\varphi). \tag{5.1}$$

As before, for $\lambda = \lambda_n$ we may assume $\varphi(a) = 0$, $\varphi(b) = (2n - 1/2)\,\tilde{\pi}_p$ so that

$$\frac{2n-1}{2}\,\tilde{\pi}_p = \lambda_n^{1/p}\|uv\|_1 + \int_a^b \frac{\omega'}{\omega}\,\mathrm{s}_p(\varphi)\mathrm{c}_p(\varphi).$$

Since $\mu_n = \lambda_n^{-1/p}$ this gives

$$\mu_n = \frac{2\|uv\|_1}{(2n-1)\tilde{\pi}_p}\left(1 - \frac{2}{(2n-1)\tilde{\pi}_p}\int_a^b \frac{\omega'}{\omega}\mathrm{s}_p(\varphi)\mathrm{c}_p(\varphi)\right)^{-1}.$$

Presently we shall show that the integral tends to zero as $n \to \infty$, so we obtain the following theorem.

Theorem 5.1. *Suppose* $\log(|v|^p/|u|^{p'})$ *is absolutely continuous. Then the singular values of T satisfy*

$$\mu_n = \frac{2\|uv\|_1}{(2n-1)\tilde{\pi}_p} + \mathrm{o}(n^{-2}) \quad \textit{as } n \to \infty, \tag{5.2}$$

Remark 5.2. It is not hard to see that if $\log(|v|^p/|u|^{p'})$ is just of bounded variation, then the error term is $\mathcal{O}(n^{-2})$.

Remark 5.3. It is worth remarking that the condition of absolute continuity for $\log(|v|^p/|u|^{p'})$ is invariant under natural transformations of the operator T, so called *isometric Liouville transforms*, which map $L^p(c, d) \ni \tilde{f} \mapsto f \in L^p(a, b)$ through setting

$$f(x) = h(x)\tilde{f}(t(x)).$$

Here t is an absolutely continuous, strictly monotone function defined on $[a, b]$ with range $[c, d]$, and h is measurable and such that $t' = |h|^p$. This makes the map isometric and surjective which is easily verified. Conjugating T by such a map changes T into an operator of the same form, and $\log(|v|^p/|u|^{p'})$ is absolutely continuous precisely if the analogous function for the transformed operator is.

To prove the theorem we make use of the following lemmas, the first of which is a simple generalisation of the Riemann–Lebesgue lemma.

Lemma 5.4. *Suppose J is a real interval, that $f \in L^\infty(\mathbb{R})$ is a periodic function with average 0 and that $g \in L^1(J)$. Then $\int_I g(x)f(x\xi)\,\mathrm{d}x \to 0$ as $\xi \to \pm\infty$, uniformly for subintervals $I \subset J$.*

Proof. First assume that g is the characteristic function of a compact subinterval $[a, b]$ of J. If $[c, d] = [a, b] \cap I$ we then have

$$\int_I g(x)f(x\xi)\,\mathrm{d}x = \int_c^d f(x\xi)\,\mathrm{d}x = \frac{1}{\xi}\int_{c\xi}^{d\xi} f,$$

so that $|\int_I g(x)f(\xi x)\,\mathrm{d}x| \leqslant \frac{1}{|\xi|}\|f\|_1$, where the last norm is taken only over a period interval, since f has average 0. Thus the lemma is true for characteristic functions of compact subintervals, and hence for step functions. Now assume just that g is integrable, and let h be a step function. Then

$$\left|\int_I g(x)f(x\xi)\,\mathrm{d}x\right| \leqslant \left|\int_I h(x)f(x\xi)\,\mathrm{d}x\right| + \|f\|_\infty \int_J |g-h|,$$

from which the lemma follows, since the first term tends to 0 uniformly with respect to I. Thus

$$\limsup_{|\xi|\to\infty} \sup_{I\subset J}\left|\int_I g(x)f(x\xi)\,\mathrm{d}x\right| \leqslant \|f\|_\infty \int_J |g-h|,$$

which can be made arbitrarily small by choice of h, since step functions are dense in $L^1(J)$. $\square$

Lemma 5.5. *Suppose f is a Lipschitz-continuous periodic function with average 0, that $g \in L^1(a, b)$ and that*

$$\varphi(x) = x\xi + \int_a^x g(y)f(\varphi(y))\,\mathrm{d}y.$$

Then $\int_a^x g(y)f(\varphi(y))\,\mathrm{d}y \to 0$ uniformly for $x \in [a, b]$ as $\xi \to \pm\infty$.

Proof. Write $F(x) = |\int_a^x g(y)f(\varphi(y))\,\mathrm{d}y|$ and let L be the Lipschitz constant for f. We then have

$$|f(\varphi(y)) - f(y\xi)| \leqslant LF(y),$$

so that

$$F(x) \leqslant \left|\int_a^x g(y)f(y\xi)\,\mathrm{d}y\right| + L\int_a^x |g|F.$$

Thus $F(x) \leqslant \sup\limits_{a<t<b} |\int_a^t g(y)f(y\xi)\,\mathrm{d}y| \exp(L\int_a^b |g|)$ by the Gronwall lemma. Thus we get the desired conclusion from Lemma 5.4. $\square$

Proof of Theorem 5.1. Integrating (5.1) we obtain

$$\varphi(x) = \lambda^{1/p}\int_a^x |uv| + \int_a^x \frac{\omega'}{\omega}\mathrm{s}_p(\varphi)\mathrm{c}_p(\varphi),$$

which after the change of variable $t=\int_a^x |uv|$ gives

$$\varphi(x(t))=\lambda^{1/p}\,t+\int_0^t \frac{\dot{\omega}}{\omega}\,\mathrm{s}_p(\varphi)\mathrm{c}_p(\varphi).$$

Thus, to prove the theorem using Lemma 5.5 we only have to verify that the function $f(x)=\mathrm{s}_p(x)\mathrm{c}_p(x)$ is Lipschitz continuous and periodic with average 0. We have

$$f'(x)=|\mathrm{c}_p(x)|^{p'}-|\mathrm{s}_p(x)|^p$$

which is clearly bounded by p, so f is Lipschitz with constant p. Clearly f is periodic with period $2\tilde{\pi}_p$ (actually, $\tilde{\pi}_p$), and since it is an odd function it has average 0. The theorem follows. □

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